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NEARAFFINE PLANES

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Nearaffine planes*)

by

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ABSTRACT

In this paper we develop a theory for nearaffine planes analogous to the theory of ordinary affine translation planes. In a subsequent paper we shall use this theory to give a characterization of a certain class of Minkowski planes.

KEY WORDS & PHRASES: *Nearaffine plane, affine plane*

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*) This report will be submitted for publication elsewhere

1. INTRODUCTION

Nearaffine spaces were introduced by J. André as a generalization of affine spaces (see e.g. [1], [2], [3]). We shall restrict our attention to nearaffine spaces of dimension 2, the nearaffine planes. Our set of axioms, defining nearaffine planes is weaker than the one used by André. If however the so-called Veblen-condition is assumed to hold (see section 3), our definition coincides with the one given by ANDRÉ in [2]. Our main goal will be to generalize the theory of translation planes to the case of nearaffine planes. In a second paper we shall show the relationship between certain nearaffine planes and Minkowski planes.

In section 2 we give the definition of a near affine plane and some basic results. Section 3 is devoted to the so-called Veblen-axiom. In section 4 we consider automorphisms of nearaffine planes, in particular translations and dilatations. In section 5 we show that translations exist whenever a certain Desarguers configuration holds. In section 6 we give an algebraic representation for nearaffine translation planes. Section 7 contains some information on the relationship with Latin squares. Finally in section 8, we give a construction of a class of nearaffine planes. More detailed information, especially on the construction of nearaffine planes, can be found in [12].

2. DEFINITION AND BASIC RESULTS

Let X be a nonempty set of elements called *points*, L a set of subsets of X called *lines*. Let $\sqcup$ be an operation called *join* mapping the ordered pairs (x,y), $x,y \in X$, $x \neq y$, onto L (the join from x to y is denoted by $x \sqcup y$), and $\parallel$ an equivalence relation called *parallelism* on L (ℓ parallel to m is denoted by $\ell \parallel m$).

We say that $(X,L,\sqcup,\parallel)$ is a *nearaffine plane* if the following three groups of axioms are satisfied.

Axioms on Lines:

(L1): $x,y \in x \sqcup y$ for all $x,y \in X$, $x \neq y$.

(L2): $z \in x \sqcup y \setminus \{x\} \Longleftrightarrow x \sqcup y = x \sqcup z$ for all $x,y,z \in X$, $x \neq y$.

(L3): $x \sqcup y = y \sqcup x = x \sqcup z \Rightarrow x \sqcup z = z \sqcup x$ for all $x,y,z \in X$, $y \neq x \neq z$.

The point x is called a *basepoint* of the line $x \sqcup y$. It is not difficult to show the following proposition (see [2]).

PROPOSITION 2.1. *The following are equivalent.*

(i) $x \sqcup y$ *has a basepoint* $\neq x$,

(ii) *each point of* $x \sqcup y$ *is a base point of* $x \sqcup y$,

(iii) $x \sqcup y = y \sqcup x$.

Therefore we may define: a line $x \sqcup y$ is called *straight* iff $x \sqcup y = y \sqcup x$. The set of all straight lines is denoted by G. The lines in $L \setminus G$ are called *proper* lines.

Axioms of parallelism:

(P1): for all $\ell \in L$, $x \in X$ there exists exactly one line with base point x parallel to ℓ.
We denote this line by $(x \parallel \ell)$.

(P2): $x \sqcup y \parallel y \sqcup x$ for all $x,y \in X$, $x \neq y$.

(P3): $(g \parallel \ell) \Rightarrow \ell \in G$ for all $g \in G$, $\ell \in L$.

Axioms on richness:

(R1): There exists at least two non-parallel straight lines.

(R2): Every line ℓ meets every straight line g with $g \nparallel \ell$ in exactly one point.

We state some basic results which follow immediately from our axioms (see e.g. [2], [11]).

PROPOSITION 2.2. *Two distinct lines with the same base point have no other point in common.*

PROPOSITION 2.3. *Two distinct straight lines intersect in one point unless they are parallel in which case they are disjoint.*

<u>THEOREM 2.4</u>. *A nearaffine plane with commutative join is an affine plane.*

We shall only consider finite nearaffine planes, i.e. nearaffine planes with a finite number of points. The following result is easy to prove (see e.g. [2], [11]).

<u>PROPOSITION 2.5</u>. *All lines of a nearaffine plane have the same number of points.*

The number of points on a line, which equals the number of parallel straight lines in one equivalence class, is denoted by n and called the *order* of the nearaffine plane.

<u>PROPOSITION 2.6</u>. $|X| = n^2$.

<u>PROPOSITION 2.7</u>. *There are exactly* n+1 *lines with a given base point.*

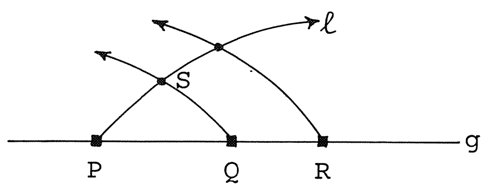
We denote by s+1 the number of equivalence classes containing straight lines. By (R1) we have $s \geq 1$.

<u>PROPOSITION 2.8</u>. *Every point is on* s+1 *straight lines,* $|G| = n(s+1)$, $|L\backslash G| = n^2(n-s)$.

3. THE VEBLEN-CONDITION

Many interesting examples of nearaffine planes (e.g. the nearaffine planes associated with Minkowski planes) satisfy the following version of the Veblen-condition (named (V') in [2]).

(V'): Let g be a straight line, P, Q, R distinct points on g, $\ell \neq g$ a line with base point P and $S \in \ell\backslash\{P\}$. Then $(R \parallel Q \sqcup S) \cap \ell \neq \emptyset$ (see fig. 1).



<u>Fig. 1</u>

Before we prove the main result on nearaffine planes which satisfy (V'), we prove a proposition valid in any nearaffine plane. Notice that until now we have not used axiom (P2) and that the proof of this proposition only requires the following weakened version of (P2) (this will be important in our paper on Minikowski planes).

(P2'): Let g and h be two distinct parallel straight lines, $x,x' \in g$ and $y,y' \in h$. Then $x \sqcup y \parallel x' \sqcup y' \Leftrightarrow y \sqcup x \parallel y' \sqcup x'$.

PROPOSITION 3.1. *Two parallel lines which have their base point on one straight line are disjoint or identical.*

PROOF. Let ℓ and ℓ' be two parallel lines with base points x and x' respectively on the straight line g. If $y \in \ell \cap \ell'$, $y \neq x,x'$ then $x \sqcup y = \ell \parallel \ell' = x' \sqcup y$, hence $y \sqcup x \parallel y \sqcup x'$ by (P2') and so $y \sqcup x = y \sqcup x'$ by (P1). Therefore $x = x'$ by (R2) and so $\ell = \ell'$ by (P1)

THEOREM 3.2. (ANDRÉ [2]). *Let* $N = (X,L,\sqcup,\parallel)$ *be a nearaffine plane satisfying* (V') *and* g *a straight line of* N. *Then the point set* X *and the line set* $L_g := \{\ell \in L \mid \ell$ *has base point on* $g\} \cup \{h \in G \mid h \parallel g\}$ *constitute an affine plane* $N_g = (X,L_g)$.

PROOF. Let $\ell,m \in L_g$, $\ell \neq m$. If $\ell \parallel m$ then $|\ell \cap m| = 0$ by 2.3 and 3.1. If $\ell \nparallel m$ then $|\ell \cap m| = 1$. This follows from (R2) if $\ell \parallel g$ or $m \parallel g$. Suppose therefore that ℓ and m have base points on g. The n line in L_g parallel to m partition X by 3.1. Hence, at least one of these lines contains a point of ℓ. Therefore, by (V') and 2.5, each of these lines, so in particular m, contains exactly one point of ℓ. Since $|L_g| = n(n+1)$ and $|\ell| = n$ for every $\ell \in L_g$ it follows from [5, result 3.2.4c, p. 139] that N_g is an affine plane. □

REMARK. Notice that two lines of N_g are parallel in N_g (i.e. disjoint) iff they are parallel in N.

4. AUTOMORPHISMS

In this section we generalize such notions as automorphism, dilatation etc. to the case of nearaffine planes. Proofs which do not differ essentially from the corresponding proofs for affine planes (see e.g. [4]) will be omitted.

DEFINITION 4.1. Let $N = (X,L,\sqcup,\|)$ and $N' = (X',L',\sqcup',\|')$ be two nearaffine planes. A bijection $\alpha: X \to X'$ is called an *isomorphism* of N and N' if

(i) $(P \sqcup Q)^\alpha = P^\alpha \sqcup' Q^\alpha$ for all $P,Q \in X$, $P \neq Q$,

and

(ii) $\ell \| m \iff \ell^\alpha \|' m^\alpha$ for all $\ell,m \in L$.

If $N = N'$, then α is called an *automorphism* of N. A permutation α of the points of N is called a *dilatation* if $P \sqcup Q \| P^\alpha \sqcup Q^\alpha$ for all $P \neq Q$.

The automorphisms of a nearaffine plane form a group A, the dilatations form a group $\mathcal{D}$.

THEOREM 4.2. $\mathcal{D} \trianglelefteq A$.

LEMMA 4.3. *Suppose* $\delta \in D$ *fixes* $P \in X$. *Then* $Q^\delta \in P \sqcup Q$ *for all* $Q \in X$, $X \neq P$.

THEOREM 4.4. *Suppose* $\delta \in D$ *fixes two distinct points* P *and* Q. *Then* $\delta = 1$.

PROOF. Take $R \in X$. If $R = P$ or $R = Q$, then $R^\delta = R$. If $R \neq P,Q$ we have by 4.3: $R^\delta \in P \sqcup R$ and $R^\delta \in Q \sqcup R$. By (R1) there is at least one straight line $g \neq P \sqcup Q$ through P, so for $R \in g$ we have $R^\delta \in (P \sqcup R) \cap (Q \sqcup R) = \{R\}$, i.e. $R^\delta = R$. For an arbitrary $R \notin g$ we replace P by a point P' in such a way that $P' \sqcup R$ is straight and Q by some point $Q \in g\backslash\{P'\}$. It follows that $R^\delta \in (P' \sqcup R) \cap (Q' \sqcup R) = \{R\}$. □

COROLLARY 4.5. *Let* $\delta_1,\delta_2 \in \mathcal{D}$ *and suppose* $P^{\delta_1} = P^{\delta_2}$, $Q^{\delta_1} = Q^{\delta_2}$ *for distinct points* P *and* Q. *Then* $\delta_1 = \delta_2$.

DEFINITION 4.6. A dilatation τ is called a *translation* if $\tau = 1$ or if $P \sqcup P^\tau \| Q \sqcup Q^\tau$ for all $P,Q \in X$. The parallel class containing $P \sqcup P^\tau$ is called the *direction* of $\tau \neq 1$. The translation τ is *straight* if $P \sqcup P^\tau$ is straight. We denote by T the set of all translations.

A translation $\tau \neq 1$ has no fixed point. Suppose $P^\tau = P$; then for any point $Q \neq P$ we have $Q^\tau \neq Q$ by 4.4 and $Q^\tau \in P \sqcup Q$ by 4.3. Hence, if $P \sqcup Q$ is straight, $Q \sqcup Q^\tau = P \sqcup Q$.

This is a contradiction since there are at least two nonparallel straight lines through P.

LEMMA 4.7. *If* $\alpha \in \mathcal{A}$ *and* $\tau \in \mathcal{T}$, *then* $\alpha\tau\alpha^{-1} \in \mathcal{T}$. *If in addition* $\alpha \in \mathcal{D}$ *and* $\tau \neq 1$, *then* τ *and* $\alpha\tau\alpha^{-1}$ *have the same direction.*

THEOREM 4.8. *Let* C *be a parallel class consisting of straight lines and* $\mathcal{T}(C) := \{\tau \in \mathcal{T} \mid \tau$ *has direction* $C\} \cup \{1\}$. *Then* $\mathcal{T}(C) \trianglelefteq D$.

LEMMA 4.9. *Let* C *and* D *be two distinct parallel classes consisting of straight lines. Then* $\sigma\tau = \tau\sigma$ *for all* $\sigma \in \mathcal{T}(C)$, $\tau \in \mathcal{T}(D)$.

LEMMA 4.10. *Let* C *and* D *be two parallel classes containing straight lines,* $\sigma \in \mathcal{T}(C)$ *and* $\tau \in \mathcal{T}(D)$. *If* $\sigma\tau \neq 1$, *then* $\sigma\tau$ *has no fixed points.*

PROOF. If $C = D$ or if σ or $\tau = 1$, this is a consequence of 4.8. If $C \neq D$ and $\sigma, \tau \neq 1$, then $P^{\sigma\tau} = P$ for some $P \in X$ implies $P \sqcup P^{\sigma} \in C$, $P \sqcup P^{\tau^{-1}} \in D$, $P^{\sigma} = P^{\tau^{-1}}$, a contradiction. □

For nearaffine planes the product of two translations need not be a translation. For straight traslations the following theorem holds.

THEOREM 4.11. *Let* C, D *and* E *be three distinct parallel classes consisting of straight lines. Suppose* $\rho \in \mathcal{T}(C)$, $\sigma \in \mathcal{T}(D)$, $\tau \in \mathcal{T}(E)$ *and* $P \in X$ *satisfy* $P^{\rho\sigma} = P^{\tau}$. *Then* $\rho\sigma = \tau$.

PROOF. If $\tau = 1$, then $P^{\rho\sigma} = P$, hence $\rho\sigma = 1$ by 4.10. If $\tau \neq 1$, then $P^{\tau} \neq P$. From 4.9 it follows that $(P^{\tau})^{\tau} = (P^{\rho\sigma})^{\tau} = (P^{\tau})^{\rho\sigma}$. Hence, $\tau = \rho\sigma$ by 4.5. □

THEOREM 4.12. *Let* C *and* D *be two distinct parallel classes consisting of straight lines with* $|\mathcal{T}(C)| = |\mathcal{T}(D)| = n$. *Then*

$$\mathcal{T} \subseteq \langle\mathcal{T}(C), \mathcal{T}(D)\rangle = \mathcal{T}(C)\mathcal{T}(D).$$

If in addition $\mathcal{T}(C)$ *and* $\mathcal{T}(D)$ *are Abelian, then* $\mathcal{T} = \mathcal{T}(C)\mathcal{T}(D)$.

PROOF. By 4.9, $\langle\mathcal{T}(C), \mathcal{T}(D)\rangle = \mathcal{T}(C)\mathcal{T}(D)$ and $|\mathcal{T}(C)\mathcal{T}(D)| = n^2$. By 4.10, $\mathcal{T}(C)\mathcal{T}(D)$ is the Frobenius kernel of $\mathcal{D}$, hence it contains all fixed-points free dilatations. Therefore $\mathcal{T} \subseteq \mathcal{T}(C)\mathcal{T}(D)$. Suppose $\mathcal{T}(C)$ and $\mathcal{T}(D)$ are Abelian. Take $\rho \in \mathcal{T}(C)$, $\sigma \in \mathcal{T}(D)$ and $P,Q \in X$. There exist $\rho_1 \in \mathcal{T}(C)$, $\sigma_1 \in \mathcal{T}(D)$ such that $P^{\rho_1\sigma_1} = Q$. Hence,

$$P \sqcup P^{\rho\sigma} \parallel (P \sqcup P^{\rho\sigma})^{\rho_1\sigma_1} = P^{\rho_1\sigma_1} \sqcup P^{\rho_1\sigma_1\rho\sigma} = Q \sqcup Q^{\rho\sigma},$$

i.e. $\rho\sigma \in \mathcal{T}$. □

A nearaffine plane having two distinct parallel classes C and D consisting of straight lines such that $|\mathcal{T}(C)| = |\mathcal{T}(D)| = n$ is called a *nearaffine translation plane*. Notice that this definition is consistent with the definition of translation plane.

<u>THEOREM 4.13</u>. *Let* C, D *and* E *be three distinct parallel classes consisting of straight lines. If* $|\mathcal{T}(C)| = |\mathcal{T}(D)| = n$, *then*
(a) $\mathcal{T}(E)$ *is Abelian,*
(b) $\mathcal{T}(C) \simeq \mathcal{T}(D)$.

<u>PROOF</u>.
(a) Let $\tau_1,\tau_2 \in \mathcal{T}(E)$. By 4.12 there exist $\rho_1 \in \mathcal{T}(C)$, $\sigma_1 \in \mathcal{T}(D)$ such that $\tau_1 = \rho_1\sigma_1$. By 4.9,

$$\tau_1\tau_2 = \rho_1\sigma_1\tau_2 = \tau_2\rho_1\sigma_1 = \tau_2\tau_1.$$

(b) Define the automorphism $\phi\colon \mathcal{T}(C) \to \mathcal{T}(D)$ as follows: Fix a line $g \in E$. For each $\rho \in \mathcal{T}(C)$ let $\phi(\rho) \in \mathcal{T}(D)$ be determined by $g^{\rho\phi(\rho)} = g$. □

<u>COROLLARY 4.14</u>. *If in addition to the hypothesis of* 4.13, $|T(E)| = n$, *then* $\mathcal{T}(C) \simeq \mathcal{T}(D) \simeq \mathcal{T}(E)$ *and these groups are Abelian.*

So far we have not used (P2) in this section. Using (P2) it is possible to prove the following theorem.

<u>THEOREM 4.15</u>. *The order* n *of a nearaffine translation plane is odd or a plane of* 2.

<u>PROOF</u>. Suppose n is even and let C and D be two distinct parallel classes

consisting of straight lines such that $|\mathcal{T}(C)| = |\mathcal{T}(D)| = n$. There exists $\rho \in \mathcal{T}(C)$ such that $\rho^2 = 1$, $\rho \neq 1$. Take $\sigma \in \mathcal{T}(D)$ and $P \in X$. Then,

$$P \sqcup P^{\rho\sigma} \| P^{\rho\sigma^{-1}} \sqcup (P^{\rho\sigma})^{\rho\sigma^{-1}} = P^{\rho\sigma^{-1}} \sqcup P \| P \sqcup P^{\rho\sigma^{-1}}.$$

Therefore $P^{\rho\sigma}, P^{\rho\sigma^{-1}} \in P \sqcup P^{\rho\sigma} \notin D$. Since $P^{\rho\sigma^{-1}}$ and $P^{\rho\sigma} = (P^{\rho\sigma^{-1}})^{\sigma 2}$ are on the same straight line of D it follows that $P^{\rho\sigma^{-1}} = P^{\rho\sigma}$, i.e. $\sigma^2 = 1$. Hence, $\mathcal{T}(D)$ is an (elementary Abelian) 2-group. □

5. A DESARGUES CONFIGURATION

Let $N = (X, L, \sqcup, \|)$ be a nearaffine plane and C a parallel class consisting of straight lines. Consider the following condition (cf. [2], [3]).

(D1): Little Desargues configuration. If $P, P', Q, Q', R, R' \in X$ are distinct points such that $P \sqcup P'$, $Q \sqcup Q'$, $R \sqcup R'$ are distinct lines of C, then $P \sqcup Q \| P' \sqcup Q'$ and $P \sqcup R \| P' \sqcup R'$ imply $Q \sqcup R \| Q' \sqcup R'$ (see figure 2).

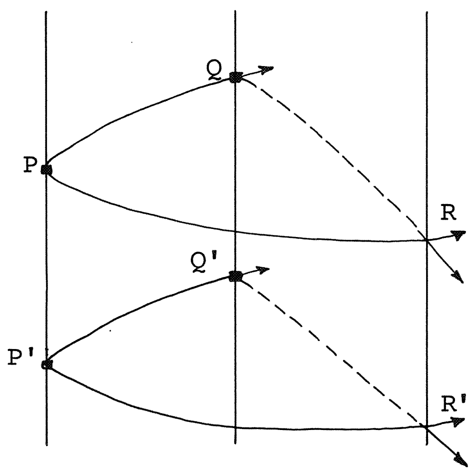


Fig. 2

Analogous to the situation for affine planes, the validity of (D1) is seen to be equivalent to the existence of all possible translations with direction C.

THEOREM 5.1. C *satisfies* (D1) $\Longleftrightarrow$ $|\mathcal{T}(C)| = n$.

The following theorem will be useful in our paper on Minkowski planes. Again notice that we only make use of (P2').

THEOREM 5.2. *Let* $N = (X, L, \sqcup, \|)$ *be a nearaffine plane in which the Veblen-condition holds, and let* C *be a parallel class of straight lines. Then (using the notation of* 3.2*),* C *satisfies* (D1) *in* $N \Longleftrightarrow$ C *satisfies* (D1) *in* N_g *for all* $g \in C$.

PROOF. $\Rightarrow$: Every translation of N with direction C is easily seen to induce a translation of N_g with direction C for every $g \in G$.

$\Leftarrow$: Let P, P', Q, Q', R, R' be distinct points such that $P \sqcup P'$, $Q \sqcup Q'$, and $R \sqcup R'$ are distinct straight lines of C and such that $P \sqcup Q \| P' \sqcup Q'$, $P \sqcup R \| P' \sqcup R'$. Let S (resp. S') be the base point of the line in $N_{P \sqcup P'}$ passing through Q and R (resp. Q' and R'), (see figure 3). Application of (D1) in $N_{P \sqcup P'}$ yields $S \sqcup Q \| S' \sqcup Q'$.

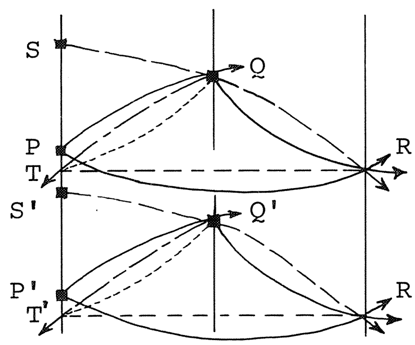


Fig. 3.

Let D be a parallel class of straight lines different from C, and let T (resp. T') be the point of intersection of $P \sqcup P'$ and the straight line of D passing through R (resp. R'). Application of (D1) in $N_{P \sqcup P'}$ to the triangles TQR and T'Q'R' yields $T \sqcup Q \parallel T' \sqcup Q'$, hence $Q \sqcup T \parallel Q' \sqcup T'$. Finally apply (D1) in $N_{Q \sqcup Q'}$ to the triangle TQR and T'Q'R' to obtain $Q \sqcup R \parallel Q' \sqcup R'$.

□

6. ALGEBRAIC REPRESENTATION

In this section an algebraic representation is given of the nearaffine translation planes. The tedious but straightforward proofs are omitted. For details see [12].

Let G and G' be two groups of order n written additively. We do not assume that G or G' is Abelian or that $G \simeq G'$ (although the same symbol + is used for addition in both groups). Let F be a set of (n-1) mappings $f_i: G \to G'$, $i = 1,\dots,n-1$, such that the following conditions are satisfied.

(i) f_i is a bijection for all $i = 1,\dots,n-1$.

(ii) $f_i(0) = 0$ for all $i = 1,\dots,n-1$.

(iii) $f_i(\alpha) = -f_i(-\alpha)$ for all $i = 1,\dots,n-1$, $\alpha \in G$.

(iv) $f_j(\alpha) \neq f_j(\alpha)$ for $1 \le i < j \le n-1$, $\alpha \in G\setminus\{0\}$.

(v) For all $i = 1,\dots,n-1$ either,

$$\forall_{\alpha\in G\setminus\{0\}} \exists_{\beta\in G} \, [f_i(\alpha+\beta) \neq f_i(\alpha) + f_i(\beta)]$$

or

$$\forall_{\alpha,\beta\in G} \, [f_i(\alpha+\beta) = f_i(\alpha) + f_i(\beta)]$$

and $f_i - f_j$ is a bijection for $j = 1,\dots,n-1$, $j \neq i$.

Given such a set of mappings F it is possible to construct a nearaffine translation plane in the following way. Put $X := G \times G'$. For $x,y \in X$, $x = (\xi,\xi')$, $g = (\eta,\eta')$, $x \neq y$, define:

$$x \sqcup y := \begin{cases} \{(\xi,\alpha') \mid \alpha' \in G'\} & \text{if } \xi = \eta, \\ \{(\alpha,\xi') \mid \alpha \in G\} & \text{if } \xi' = \eta', \\ \{((\xi+\alpha,\ \xi' + f_i(\alpha) \mid \alpha \in G\} & \text{if } \xi \neq \eta,\ \xi' \neq \eta' \text{ and} \\ & \quad f_i(-\xi+\eta) = -\xi' + \eta'. \end{cases}$$

The line set L is just the set of all $x \sqcup y$, $x \neq y$. For any line $\ell = x \sqcup y$ we let $d(\ell) \in \{0,1,\ldots,n-1,\infty\}$ be determined by

$$d(\ell) := \begin{cases} \infty & \text{if } \xi = \eta, \\ 0 & \text{if } \xi' = \eta', \\ i & \text{if } \xi' \neq \eta',\ \xi' \neq \eta' \text{ and } f_i(-\xi+\eta) = -\xi' + \eta'. \end{cases}$$

Notice that $d(\ell)$ only depends on ℓ and not on the special choice of x and y. Define parallelism by

$$\ell \parallel m :\iff d(\ell) = d(m),$$

then $N = (X,L,\sqcup,\parallel)$ is a nearaffine translation plane. Conversely, every nearaffine translation plane can be described in this way. The parallel classes $C_0 := \{\ell \in L \mid d(\ell) = 0\}$, $C_\infty := \{\ell \in L \mid d(\ell) = \infty\}$ consist of straight lines. For each $\alpha \in G$, the mapping $(\xi,\xi') \to (\alpha+\xi,\xi')$ is a translation with direction C_0. For each $\alpha' \in G'$, the mapping $(\xi,\xi') \to (\xi,\alpha'+\xi')$ is a translation with direction C_∞. For $i = 1,\ldots,n-1$, $C_i := \{\ell \in L \mid d(\ell) = i\}$ consists of straight lines iff i satisfies the second alternative of (v).

The Veblen-condition (V') is satisfied if for $1 \leq i < j \leq n-1$,

(a) $f_i - f_j : G \to G'$ is a bijection,
(b) $f_i^{\leftarrow} - f_j^{\leftarrow} : G' \to G$ is a bijection,
(c) for all $k \in \{1,\ldots,n-1\}$ which satisfy the second alternative of (v) and for all $\gamma \in G$ there is a unique solution α of $f_k(\gamma) = f_j(\gamma+\alpha) - f_i(\alpha)$.

7. NEARAFFINE PLANES AND LATIN SQUARES

It is well known that the existence of an affine plane of order n is equivalent to the existence of n-1 mutually orthogonal Latin squares (M.O.L.S.) of order n (see [5]). For nearaffine plane the following result holds.

THEOREM 7.1. *If N is a nearaffine plane of order* n *with* s+1 *parallel classes containing straight lines* (s < n), *then there exist* s M.O.L.S. *of order* n.

PROOF. The n(s+1) lines in the s+1 parallel classes consisting of straight lines together with n lines from a parallel class consisting of proper lines, all having their base points on a fixed straight line, constitute an $(s+2)$-net of order n. This is equivalent to the existence of s M.O.L.S. of order n (see e.g. [5]). $\square$

Let N be an integer, $N \geq 2$, and suppose N has prime decomposition $N = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$. Define

$$s(N) := \min_{1 \leq i \leq k} p_i^{\alpha_i} - 1.$$

It is well known (see e.g. [5]) that there exist at least $s(N)$ M.O.L.S. of order N (the so-called MacNeish bound). The following theorem shows therefore that, as far as nearaffine translation planes are concerned, we cannot hope for interesting applications of 7.1.

<u>THOEREM 7.2</u>. *Let $\mathcal{N}$ be a nearaffine translation plane of order* n *with* $s+1$ *parallel classes containing straight lines. Then* $s \leq s(n)$.

<u>PROOF</u>. Notice that $s-1$ of the f_i's associated with $\mathcal{N}$, say $f_1, f_2, \ldots, f_{s-1}$, satisfy the second alternative of (v) of section 6. Put $\phi_i := f_1^{\leftarrow} \circ f_i$, $i = 1,2,\ldots,s$. Then $\phi_i - \phi_j \colon G \to G$ is a permutation of the elements of G, $1 \leq i < j \leq s$. Hence, the Latin squares $A^{(i)} = [a_{x,y}^{(i)}]$ defined by

$$a_{x,y}^{(i)} := \phi_i(x) + y, \qquad i = 1,\ldots,s, \quad x,y \in G,$$

are mutually orthogonal. Since $\phi_1, \phi_2, \ldots, \phi_{s-1}$ are automorphisms of G it follows by a theorem of H.B. MANN (see [6] or [9]) that $s-1 \leq s(n)$. Suppose $s-1 = s(n) = p^{\alpha}-1$, p a prime, $\alpha \in \mathbb{N}$. It follows from the proof of Mann's theorem that the elements $\neq 0$ of a Sylow p-subgroup P of G are all in different conjugacy classes. Thus $-y+x+y \in P \Rightarrow -y+x+y = x$ for all $x \in P$, $y \in G$. In particular, if $y \in N_G(P)$ then $x+y = y+x$ for all $x \in P$, i.e. $P \leq Z(N_G(P))$. By a theorem of BURNSIDE (see [7] or [8]), G contains a normal p-complement N. Since $|G \backslash N|$ and $|N|$ are coprime, N is a characteristic subgroup of G. Thus the rows and columns of $A^{(1)}, \ldots, A^{(s-1)}$ which correspond to the elements of N yield mutually orthogonal Latin subsquares of order n/p^{α}. By a theorem of PARKER (see [10]) such a set of $s-1$ M.O.L.S. cannot be extended to a set of s M.O.L.S., a contradiction. Hence $s-1 < s(n)$, i.e. $s \leq s(n)$. $\square$

8. CONSTRUCTION OF NEARAFFINE PLANES

Using the representation of nearaffine translation planes of section 6, we treat a special case of the more general construction described in [12]. The nearaffine planes thus obtained turn out to be associated with certain Minkowski planes. Let p be a prime, h a positive integer and $n = p^h$. For the groups G and G' of section 6 we take the additive group of GF(n). Fix an automorphism ϕ of GF(n), and for each $a \in GF(n)^*$ define $f_a: GF(n) \to GF(n)$ by $f_a(0) := 0$ and

$$f_a(x) := ax^{-1}, \quad x \in GF(n)^*, \text{ if a is a square,}$$

$$f_a(x) := a(x^{-1})^{\phi}, \quad x \in GF(n)^*, \text{ if a is a nonsquare.}$$

The set $\mathcal{F} := \{f_a \mid a \in GF(n)^*\}$ is easily seen to satisfy the properties (i),...,(v) of section 6. The corresponding nearaffine plane is of order n, and s = 1. It is also not hard to show that the Veblen-condition holds in these nearaffine planes.

REFERENCES

[1] ANDRÉ, J., *Eine Kennzeichung der Delatationsgruppen desarguesscher affiner Räume als Permutationsgruppen,* Arch. Math. 25 (1974), 411-418.

[2] ANDRÉ, J., *Some new results on incidence structures,* Atti dei Convegni Lincei 17, Colloquio Internazionale sulle teorie combinatori II (1976), 201-222.

[3] ANDRÉ, J., *On finite non-commutative affine spaces,* in Combinatorics part I, (M. Hall and J.H. van Lint editors), Mathematical Centre Tracts 55 (1974), 60-107.

[4] ARTIN, E., *Geometric Algebra,* Interscience, New York - London (1957).

[5] DEMBOWSKI, P., *Finite Geometries,* Springer-Verlag, Berlin - Heidelberg - New York (1968).

[6] DENES, J. & A.D. KEEDWELL, *Latin squares and their applications,* The English Universities Press Ltd., London (1974).

[7] HALL, M., *The theory of groups,* MacMillan, New York (1959).

[8] HUPPERT, B., *Endliche Gruppen I,* Springer-Verlag, Berlin - Heidelberg - New York (1967).

[9] MANN, H.B., *The construction of orthogonal Latin-squares,* Ann. Math. Stat. (1943), 401-414.

[10] PARKER, E.T., *Nonextendibility conditions on mutually orthogonal Latin-squares,* Proc. Amer. Math. SOc. 13 (1963), 219-221.

[11] SCHOOT, J. VAN DER & H. WILBRINK, *Nearaffine planes I,* Indag. Math. 37 (1975), 137-143.

[12] WILBRINK, H., *Nearaffine planes and Minkowski planes,* Master's thesis, Techn. Univ. Eindhoven (1978).